LAGRANGIAN FORMULATION OF NEUTRINO OSCILLATION

Ong Jian Fuh[1] and Ithnin Abdul Jalil[2]

[1, 2] Department of Physics, Faculty of Science, University of Malaya, 50603 Kuala Lumpur, Malaysia

*Email address: [1]*thoms2543@yahoo.com, *[2]*ithninaj@um.edu.my

ABSTRACT

Neutrino is one of the nuclear particles that are necessary for the correct description of nuclear beta decay. The standa rd idea is that it is a massless neutral particle and its existence was postulated in order to save the conservation of ener gy principle. This particle was later detected experimentally and it is now known that neutrino has mass. The problem o f astrophysical neutrino detection has produced a new phenomenon of neutrino oscillation where the three neutrino flav ours can oscillate between themselves. This paper studies the two component neutrino oscillation problem. We study the neutrino oscillation by using the Lagrangian formulation. In our study, we assume that the neutrinos are produced as n eutrino mass eigenstate and propagate in the vacuum in the superposition of two neutrino flavour state. The Lagrangia n for neutrinos with their mass and the oscillation terms were obtained. By using the mass matrix in the Lagrangian, we formulate the time evolution operator in the interaction picture. The neutrino oscillation probability obtained by using t he Lagrangian formulation have the same result with the one obtained by using quantum mechanics formulation. This st udy hopes to gain some deeper understanding into the behaviour of neutrino beyond the Standard Model.

ABSTRAK

Neutrino merupakan salah satu zarah nuklear yang diperlukan untuk keterangan yang betul dalam pereputan beta nukl ear. Idea standard adalah bahawa neutrino merupakan zarah neutral yang tidak berjisim dan keperluanya dipostulasik an untuk menyelamatkan hukum keabadian tenaga. Zarah ini kemudian dikesan dalam eksperimen dan kini diketahui b ahawa neutrino mempunyai jisim. Masalah penemuan neutrino dari angkasa telah memperkenalkan satu fenomena bar u iaitu ayunan neutrino di mana tiga jenis neutrino boleh menukar ganti sesama sendiri. Neutrino dua komponen dikaji dalam kajian kami.Kami mengkaji ayunan neutrino dengan menggunakan formulasi Lagrangian. Dalam kajian ini, kam i menggapkan bahawa neutrino dihasil sebagai eigenstate jisim neutrino dan disebarkan ke dalam vakum dalam superp osisi dua keadaan flavor. Lagrangian bagi neutrino yang mempunyai informasi jisim dan ayunan diperolehi. Dengan m enggunakan matrik jisim dalan Lagrangian, kami merumuskan operator evolusi masa dalam gambaran interaksi. Keba rangkalian yang diperolehi daripada formulasi lagrangian adalah sama dengan menggunakan formulasi mekanik quan tum.Penelitian ini diharapkan dapat memperolehi pemahaman yang lebih dalam tantang perilaku neutrino di luar Mod el Standard.

Keywords: Neutrino oscillation, neutrino mixing, Lagrangian formulation.

INTRODUCTION

A neutral particle was first postulated in 1930 by Pauli in order to describe the missing energy in the nuclear beta decay. This neutral particle was later named by Fermi as neutrino (Griffith, 2008). The neutrinos that are produced from the beta decay propagate in the vacuum at almost the speed of light and pass through the matter at almost no interaction. The nuclear beta decay is described by weak interaction. In particular, only lepton and quark participate in weak interaction. The weak interaction was later unified with the electromagnetic interaction to give a more complete theory called electroweak theory (Griffith, 2008). In the electroweak theory, there are possibilities for the interaction of neutrino with the electromagnetic field at higher order of interaction. In the standard model of electroweak interaction, neutrino is assumed to be massless. The zero mass of the neutrino makes the standard model as a complete theory and no further improvement is necessary (Mandl and Shaw,1986). However, this situation is no longer valid with the discovery of small non-vanishing neutrino mass (Yao, 2006). Such a new discovery leads to the new phenomena called neutrino oscillation (Bilenky and Pontecorvo, 1958). Neutrino oscillation is a phenomenon where the neutrino of one flavour changes its type into another flavour during their propagation. Neutrino oscillation is possible if the neutrino is massive and mixed. Thus, the flavour changing process implies the lepton-family-number violation in the standard model.

In the year of 1950s, the theory to describe the neutrino oscillation had been pursued vigorously by Bilenky and Pontecorvo (1978) in the framework of quantum theory in analogy with $K^0 - \overline{K^0}$ oscillation. However, the neutrino oscillation probability was carried out rigorously in the flavour state, but we have no direct method to obtain the probability from the Lagrangian of the standard electroweak model. In our study, we intend to include the neutrino oscillation into the standard model so that it can completely describe all the interaction of particles within a single Lagrangian.

The plan of this paper is as follow. We begin with neutrino oscillation formulation in the framework of quantum theory. We then present our work by assuming that the neutrinos are produced in finite mass state and propagating in the vacuum in the combination of two neutrino flavour state and oscillate among them. We then formulate the new Lagrangian by inserting the neutrino mixing into the theory. By using the mass matrix in the Lagrangian, we obtain the time evolution operator in the interaction picture to produce the neutrino oscillation probability.

STANDARD THEORY OF NEUTRINO OSCILLATION

In the standard theory of neutrino oscillation, the neutrino flavour state is the superposition of three neutrino mass eigenstates (Koh *et. al.*, 1991).

$$|\nu_\alpha\rangle = \sum_k U^*_{\alpha k}|\nu_k\rangle \tag{1}$$

where $\alpha = e, \mu, \tau$; $k = 1, 2, 3$ and U is the 3×3 mixing matrix. The neutrino flavour state is assumed to have a definite momentum $\boldsymbol{p}$ and the mass eigenstates have different energies which in relativistic approximation is given by

$$E_k = \sqrt{p^2 + m_k^2} \approx p + \frac{m_k^2}{2p} \tag{2}$$

such that the massive neutrino states $|\nu_i\rangle$ are eigenstates of Hamiltonian (Giunti *et.al.*, 2007),

$$H|\nu_k\rangle = E_k|\nu_k\rangle \tag{3}$$

The mass eigenstate satisfies the Schrödinger-like equation

$$i\frac{\partial}{\partial t}|\nu_k(t)\rangle = H|\nu_k(t)\rangle \tag{4}$$

and has a plane wave solution

$$|\nu_k(t)\rangle = e^{-iE_k t}|\nu_k\rangle \tag{5}$$

Thus, the flavour state evolves in time as

$$|\nu_\alpha(t)\rangle = \sum_k e^{-iE_k t}U^*_{\alpha k}|\nu_k\rangle \tag{6}$$

The mass eigenstate can be expressed in term of flavour states by inverting equation (1)

$$|\nu_k\rangle = \sum_\alpha U_{\alpha k}|\nu_\alpha\rangle \tag{7}$$

Substitute equation (7) into equation (6), we get

$$|\nu_\alpha(t)\rangle = \sum_\beta \left(\sum_k U^*_{\alpha k}e^{-iE_k t}U_{\beta k}\right)|\nu_\beta\rangle \tag{8}$$

In order to obtain the neutrino oscillation probability, we multiply equation (8) by $\langle\nu_\beta|$ and take the modulus square, we have

$$P(\nu_\alpha \to \nu_\beta; t) = \left|\langle\nu_\beta|\nu_\alpha(t)\rangle\right|^2$$

$$= \sum_{k,j} U^*_{\alpha k}\, U_{\beta k} U_{\beta j} U^*_{\alpha j} e^{-i(E_k - E_j)} t \tag{9}$$

For two component of neutrino case, the probability is therefore

$$P_{\nu_e \leftrightarrow \nu_\mu} = \frac{1}{2}\sin^2 2\theta\, [1 - \cos\left(\frac{m_1^2 - m_2^2}{2E}\right) T] \tag{10}$$

LAGRANGIAN WITH NEUTRINO OSCILLATION

The two component neutrino flavour states are assumed to be the combination of neutrino mass eigenstate as in equation (1)

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{11}$$

where θ is the mixing angle, ν_1 and ν_2 are neutrino mass eigenstates with masses m_1 and m_2 respectively.

In our work, we assume that the neutrinos are produced as neutrino mass eigenstate rather than neutrino flavour state (Bilenky, 2005).

$$a \longrightarrow b + l + \nu_i, \quad (l = e, \mu \ \ i = 1{,}2) \tag{12}$$

The production of electron, e is coupled with ν_1 and of the muon, μ with ν_2. The Dirac Lagrangian is,

$$\mathcal{L}_0 = \bar{\nu}_1(i\gamma^\alpha \partial_\alpha - m_1)\nu_1 + \bar{\nu}_2(i\gamma^\alpha \partial_\alpha - m_1)\nu_2 \tag{13}$$

where $\alpha = 0,1,2,3$ are the four vector indices. With the neutrino mixing,

$$\mathcal{L}_0 = \bar{\nu}_e(i\gamma^\alpha \partial_\alpha - m_{ee})\nu_e + \bar{\nu}_\mu(i\gamma^\alpha \partial_\alpha - m_{\mu\mu})\nu_\mu + m_{e\mu}(\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e) \tag{14}$$

where,

$$\begin{aligned} m_{ee} &= m_1 \cos^2\theta + m_2 \sin^2\theta \\ m_{\mu\mu} &= m_1 \sin^2\theta + m_2 \cos^2\theta \\ m_{e\mu} &= (m_1 - m_2)\sin\theta\cos\theta \end{aligned} \tag{15}$$

Clearly m_{ee} and $m_{\mu\mu}$ are the bare masses of electron and muon neutrinos (Bilenky and pontecorvo, 1978). The dependence of electron and muon neutrino masses on m_1 and m_2 and the mixing angle, θ show that the neutrino mass is not definite and varies in certain range. The third term in the Lagrangian of equation (4) shows that the neutrinos oscillate between two flavour states. The flavour changes are due to the mass difference between two neutrino mass eigenstate, $m_{e\mu}$.

NEUTRINO OSCILLATION PROBABILITY

In the interaction picture, the Hamiltonian of a system is split in to two parts,

$$H = H_0 + H_I \tag{16}$$

where H_0 is the free-field Hamiltonian and H_I describe the interaction between two field (Mandl and Shaw,1986). The field in the interaction picture is time dependent can be written as

$$\Phi(t) = U(t, t_0)\Phi(t_0) \tag{17}$$

where $U(t, t_0)$ is the time evolution operator which bring the field from initial state at time t_0 to final state at time t. The time evolution operator satisfies the Schrödinger-like equation

$$i\frac{\partial}{\partial t}\, U(t, t_0) = H_I U(t, t_0) \tag{18}$$

with the initial condition, $U(t_0, t_0) = 1$. The solution to the equation (18) is

$$U(t, t_0) = \exp[-i \int H_I dt] \tag{19}$$

and can be written in terms of Hamiltonian density, $\mathcal{H}_I$ as

$$U(t, t_0) = \exp[-i \int \mathcal{H}_I d^4x] \tag{20}$$

In the case of neutrino, the Lagrangian density can be written in matrix form as

$$\mathcal{L} = \overline{\Psi}_f i\gamma^\alpha \partial_\alpha \Psi_f + \overline{\Psi}_f M \Psi_f \tag{21}$$

where $\Psi_f = \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}$ is the two-component neutrino flavour state, and $M = UM_0U^{-1}$ is the neutrino mass matrix.

In order to describe the neutrino oscillation by interaction picture, we split the Lagrangian density in equation (16) into the free-field Lagrangian and interaction Lagrangian. The first term in equation (16) is taken to be free-field Lagrangian which is the kinetic term. The mass term is taken to be the interaction Lagrangian.

The Hamiltonian density is defined by (Mandl and Shaw, 1986)

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}}\dot{\Phi} - \mathcal{L} \tag{22}$$

The interaction Hamiltonian density does not depend on $\dot{\Phi}$, so that the Hamiltonian density is $\mathcal{H}_I = -\mathcal{L}_I$. Hence,

$$\mathcal{L}_I = \overline{\Psi}_f UM_0U^{-1}\Psi_f$$

$$= \begin{pmatrix} \nu_e & \nu_\mu \end{pmatrix} \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} \tag{23}$$

By applying equation (22) and (23) into (20) we get the evolution operator in term of Lagrangian density. The time evolution operator is expanded into series up to second term. By using the normalization condition and the unitary relation of the mixing matrix, $UU^{-1} = 1$ we get

$$U(t,t_0) = \exp[-i \int \overline{\Psi}_f U M_0 U^{-1} \Psi_f d^4x]$$

$$= 1 + (-i) \iint dt d^3\vec{x}\, \overline{\Psi}_f U M_0 U^{-1} \Psi_f + \frac{(-i)^2}{2} \iint dt d^3\vec{x}\, \overline{\Psi}_f U M_0 U^{-1} \Psi_f \iint dt d^3\vec{x}\, \overline{\Psi}_f U M_0 \ + \cdots$$

$$= \int d^3\vec{x} \overline{\Psi}_f \exp[-i \int U M_0 U^{-1} dt] \Psi_f$$

$$= \exp[-i \int U M_0 U^{-1} dt]$$

$$= U \exp[-i \int M_0 dt] U^{-1}$$

$$= \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} e^{-im_1 t} & 0 \\ 0 & e^{-im_2 t} \end{pmatrix} \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta\, e^{-im_1 t} + \sin^2\theta\ e^{-im_2 t} & -sin\theta cos\theta (e^{-im_1 t} - e^{-im_2 t}) \\ -sin\theta cos\theta (e^{-im_1 t} - e^{-im_2 t}) & \cos^2\theta\, e^{-im_2 t} + \sin^2\theta\ e^{-im_1 t} \end{pmatrix} \quad (24)$$

The diagonal elements in the matrix in equation (24) are the terms for describing the survival of neutrino flavours while the off-diagonal elements are the terms responsible for the neutrino flavour changes. By taking the square of the off-diagonal elements one obtain

$$P_{\nu_e \leftrightarrow \nu_\mu} = \frac{1}{2} \sin^2 2\theta\, [1 - \cos(m_1 - m_2)t] \quad (25)$$

where t is the proper time in the neutrino frame. By Lorentz transformation

$$m_i t = m_i \gamma (T - vx)$$

$$= m_i \gamma T - m_i \gamma \vec{v}\vec{x}$$

$$= E_i T - \vec{p}\vec{x} \quad (26)$$

where T is the time in the laboratory frame, γ is relativistic constant, $\vec{v}$ is velocity of the neutrino, E_i and $\vec{p}$ are the energy and momentum in the laboratory frame respectively. Making the approximation as in equation (2) we obtain the neutrino oscillation probability

$$P_{\nu_e \leftrightarrow \nu_\mu} = \frac{1}{2} \sin^2 2\theta\, [1 - \cos\left(\frac{m_1^2 - m_2^2}{2E} T\right)] \quad (27)$$

As the neutrino travel almost at the speed of light, it is possible to do the approximation T=L and the equation is equivalent to

$$P_{\nu_e \leftrightarrow \nu_\mu} = \sin^2 2\theta \sin^2\left(\frac{m_1^2 - m_2^2}{4E} L\right) \quad (28)$$

This result obtained using Lagrangian formulation is the same with equation (10) which have been derived using quantum mechanical method.

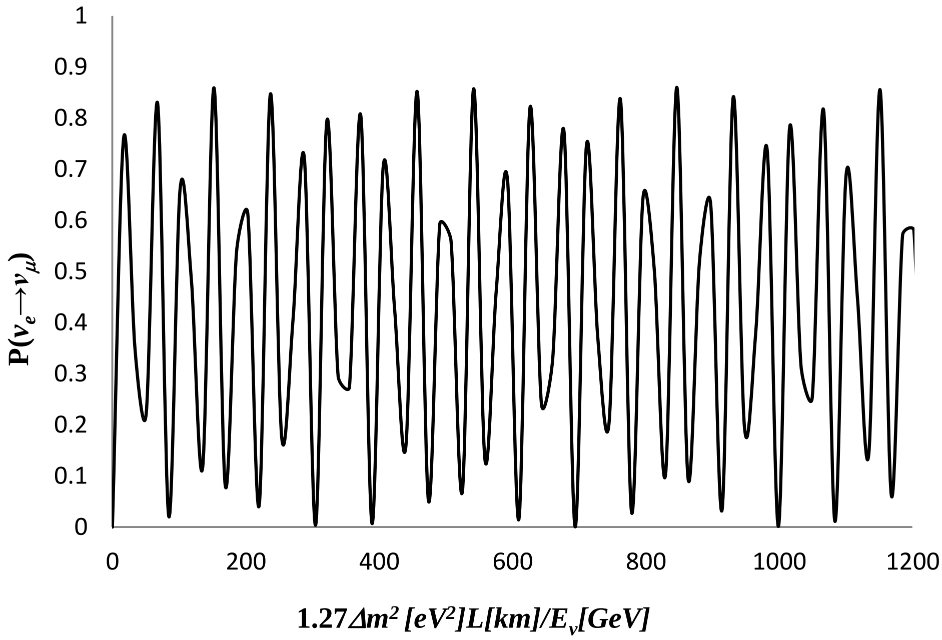

Figure. 1. Probability of neutrino oscillation as a function of 1.27□m^2 [eV^2]L[km]/E_ν[GeV] for $sin^2 2\theta = 0.86, \Delta m^2 = 8 \times 10^{-5} eV^2\ and\ E_\upsilon = 0.0006 GeV$.

In figure 1, the neutrino oscillation probability as a function of $1.27\Delta m^2$ [eV^2]L[km]/E_ν[GeV] is obtained. The mixing angle, mass square difference is base on the case of solar neutrino which has the energy of 0.6 MeV (Yao, 2006). From figure 1 we can see that the neutrino probability vary periodically along the distance from the source which mean that we may detect neutrino of different flavor along the distance. This also explain the reason that neutrino detected at the earth surface is less than the prediction from standard solar model.

CONCLUSIONS

In this work, the mass term in the Lagrangian is used to calculate the oscillation probability for two component neutrino. The oscillation probability obtained is the same with the prediction using quantum mechanical method. The method we use to obtain the probability has the similar structure to the S-matrix in quantum field theory to all orders (Peskin and Schroeder, 1995).

ACKNOWLEDGEMENT

The first author would like to thank to K. Y. Tee for fruitful discussion. He is grateful for the financial support from the Institute of Postgraduate Studies. This work was supported in part by the Institute of Research Management and Monitoring, Grant No. PS310/2009C.

REFERENCES

Bilenky, S. M., (2005), Neutrino Oscillations and Time-Energy Uncertainty Relation, hep-ph/0512215.

Griffiths, D., (2008), Introduction to Elementary Particles (2nd ed.). New York: John Wiley.

Bilenky, S. M. and Pontecorvo, B., (1976), Quark Lepton Analogy and Neutrino Oscillation, Phys.Lett.B, 61, 248.

Bilenky, S. M. and Pontecorvo, B., (1978), *Lepton Mixing and Neutrino Oscillations*, Phys. Rept. 41, 225.

Giunti, C. and Kim, C. W., (2007), Fundamental of Neutrino Physics and Astrophysics, New York: Oxford University Press.

Koh, J. K., Guinti, C. and Kim, C. W., (1991), CP Violation in the Lepton Sector and Majorana Neutrinos, J. Korean Phys. Soc., 24, 275.

Mandl, F. and Shaw. G., (1986), Quantum Field Theory, Great Britain: John Wiley.

Peskin, M. E. and Schroeder, D. V., (1995), An Introduction to Quantum Field Theory, Addison-Wesley.

Pontecorvo, B., (1958), Inverse Beta Processes and Nonconservation of Lepton Charge, JETP, 34, 247.

Yao, W-M. et al., (2006), Review of Particles Physics, J. Phys. G, 33, 472.